# Two results on ill-posed problems *†

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**Abstract**

Let $A = A^*$ be a linear operator in a Hilbert space $H$. Assume that equation $Au = f$ (1) is solvable, not necessarily uniquely, and $y$ is its minimal-norm solution. Assume that problem (1) is ill-posed. Let $f_\delta$, $||f - f_d|| \leq \delta$, be noisy data, which are given, while $f$ is not known. Variational regularization of problem (1) leads to an equation $A^*Au + \alpha u = A^*f_\delta$. Operation count for solving this equation is much higher, than for solving the equation $(A + ia)u = f_\delta$ (2). The first result is the theorem which says that if $a = a(\delta)$, $\lim_{\delta \to 0} a(\delta) = 0$ and $\lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution $u_\delta$ to equation (2), with $a = a(\delta)$, has the property $\lim_{\delta \to 0} ||u_\delta - y|| = 0$. The second result is an iterative method for stable calculation of the values of unbounded operator on elements given with an error.

## 1. Introduction

The results of this note are formulated as Theorems 1 and 2 and proved in Sections 1 and 2 respectively. For the notions, related to ill-posed problems, one may consult [1] and [2] and the literature cited there.

Let $A = A^*$ be a linear operator in a Hilbert space $H$. Assume that equation

$$Au = f, \tag{1}$$

is solvable, not necessarily uniquely, and $y$ is its minimal-norm solution, $y \perp N := \{u : Au = 0\}$. Assume that problem (1) is ill-posed. In this case small perturbations of $f$ may cause large perturbations of the solution to (1) or may throw $f$ out of the range of $A$. Let $f_\delta$, $||f - f_d|| \leq \delta$, be noisy data, which are given, while $f$ is not known. Variational regularization of problem (1) leads to an equation $A^*Au + \alpha u = A^*f_\delta$, where $A^* = A$ since we assume $A$ to be selfadjoint. Operation count for solving this equation is much higher, than for solving the equation

$$(A + ia)u = f_\delta, \tag{2}$$

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The result of this paper is the following theorem.

**Theorem 1.** *Let $A = A^*$ be a linear bounded, or densely defined, unbounded, self-adjoint operator in a Hilbert space. Assume that $a = a(\delta) > 0$, $\lim_{\delta \to 0} a(\delta) = 0$ and $\lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution $u_\delta$ to equation (2) with $a = a(\delta)$ has the property*

$$\lim_{\delta \to 0} ||u_\delta - y|| = 0. \tag{3}$$

Why should one be interested in the above theorem? The answer is: because the solution to equation (2) requires less operations than the solution of the equation $(A^*A + \alpha I)u = A^*f_\delta$ basic for the variational regularization method for stable solution of equation (1). Here $I$ is the identity operator. Also, a discretized version of (2) leads to matrices whose condition number is of the order of square root of the condition number of the matrix corresponding to the operator $A^*A + \alpha(\delta)I$, where $I$ is the identity operator.

**Proof of Theorem 1.** One has

$$||u_{a,\delta} - y|| \le ||(A + ia)^{-1}(f_\delta - f)|| + ||(A + ia)^{-1}Ay - y|| \le \frac{\delta}{a} + a||(A + ia)^{-1}y||. \tag{4}$$

Moreover,

$$\lim_{a \to 0} a^2||(A + ia)^{-1}Ay - y||^2 = \lim_{a \to 0} a^2 \int_{-\infty}^{\infty} \frac{d(E_s y, y)}{s^2 + a^2} = 0, \tag{5}$$

where we have used the spectral theorem, $E_s$ is the resolution of the identity corresponding to the selfadjoint operator $A$, and we have taken into account that

$$\lim_{a \to 0} a^2 \int_{-0}^{0} \frac{d(E_s y, y)}{s^2 + a^2} = 0$$

because $y \perp N$. From formulas (4) and (5) one concludes that if $a = a(\delta) > 0$, $\lim_{\delta \to 0} a(\delta) = 0$ and $\lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0$, then the unique solution $u_\delta$ to equation (2) with $a = a(\delta)$ satisfies equation (3).

Theorem 1 is proved. $\square$

2. **Calculation of values of unbounded operators**

Assume that $A$ is a densely defined closed linear operator in $H$. We do not assume in this Section that $A$ is selfadjoint. If $f \in D(A)$, then we want to compute $Af$ given noisy data $f_\delta$, $||f_\delta - f|| \le \delta$. Note that $f_\delta$ may not belong to $D(A)$. The problem of stable calculation of $Af$ given the data $\{f_\delta, \delta, A\}$ is ill-posed. It was studied in the literature (see, e.g., [1]) by a variational regularization method. Our aim is to reduce this problem to a standard equation with a selfadjoint bounded operator $0 \le B \le I$, and solve this equation stably by an iterative method.

Let $v = Af$. This relation is equivalent to

$$Bv = Ff, \tag{6}$$

where $B := (I+Q)^{-1}$, $F := BA$, $Q = AA^*$ is a densely defined, non-negative, selfadjoint operator, the range of $I + Q$ is the whole space $H$, and $B$ is a selfadjoint operator, $0 \leq B \leq I$, where the inequalities are understood in the sense of quadratic forms, e.g., $B \geq 0$ means $(Bg, g) \geq 0$ for all $g \in H$, and $F := (I + Q)^{-1}A$.

**Lemma 1.** (see [3]) *The operator $(I+Q)^{-1}A$, originally defined on $D(A)$, is closable. Its closure is a bounded, defined on all of $H$ linear operator with the norm $\leq \frac{1}{2}$. One has $(I + Q)^{-1}A = A(I + T)^{-1}$, where $T = A^*A$ is a non-negative, densely defined selfadjoint operator, and $||A(I + T)^{-1}|| \leq \frac{1}{2}$.*

If $f_\delta$ is given in place of $f$, then we stably solve equation (6) for $v$ using the following iterative process:

$$v_{n+1} = (I - B)v_n + Ff_\delta, \quad v_0 \perp N^*, \tag{7}$$

where $N^* := \{u : A^*u = 0\}$. Let $y$ be the unique minimal-norm solution to equation (6), $By = Ff$. Note that $y = Hy + Ff$, where $H := I - B$.

**Theorem 2.** *If $n = n(\delta)$ is an integer, $\lim_{\delta \to 0} n(\delta) = \infty$ and $\lim_{\delta \to 0}[\delta n(\delta)] = 0$, then*

$$\lim_{\delta \to 0} ||v_\delta - y|| = 0, \tag{8}$$

*where $v_\delta := v_{n(\delta)}$, and $v_n$ is defined in (7).*

**Proof of Theorem 2.** From (7) one gets $v_{n+1} = \sum_{j=0}^n H^j Ff_\delta + H^{n+1}u_0$, where $H := I - B$. One has $y = Hy + Ff$. Let $w_n := v_n - y$. Then $w_n = \sum_{j=0}^{n-1} H^j Fg_\delta + H^n w_0$, where $g_\delta := f_\delta - f$, and $w_0$ is an arbitrary element such that $w_0 \perp N^*$. Since $0 \leq H \leq I, ||F|| \leq \frac{1}{2}$, and $||g_\delta|| \leq \delta$, one gets

$$||w_n|| \leq \frac{n\delta}{2} + [\int_0^1 (1 - s)^{2n} d(E_s, w_0, w_0)]^{1/2}, \tag{9}$$

where $E_s$ is the resolution of the identity corresponding to the selfadjoint operator $B$.

If $w_0 \perp N^*$, then

$$\lim_{h \to 1} \int_h^1 (1 - s)^{2n} d(E_s, w_0, w_0) = ||Pw_0||^2, \tag{10}$$

where $P$ is the orthoprojector onto the subspace $\{u : Bu = u\} = \{u : Qu = 0\} = N^*$, and $Pw_0 = 0$ because $w_0 \perp N^*$ by the assumption. The conclusion of Theorem 2 can now be derived. Given an arbitrary small $\epsilon > 0$, find $h$ sufficiently close to 1 such that $\int_h^1 (1 - s)^{2n} d(E_s, w_0, w_0) < \epsilon$. Fix this $h$ and find $n = n(\delta)$, sufficiently large, so that $\delta n(\delta) < \epsilon$ and, at the same time, $(1 - h)^{2n(\delta)} < \epsilon$. This is possible if $\delta$ is sufficiently small, because $\lim_{\delta \to 0} n(\delta) = \infty$ and $\lim_{\delta \to 0}[\delta n(\delta)] = 0$. Then $\int_0^1 (1 - s)^{2n} d(E_s, w_0, w_0) \leq \int_0^h (1 - s)^{2n} d(E_s, w_0, w_0) + \int_h^1 (1 - s)^{2n} d(E_s, w_0, w_0) < \epsilon$, and inequality (9) shows that (8) holds. Theorem 2 is proved. $\quad\square$.

**Remark 1.** It is not possible to estimate the rate of convergence in (8) without making additional assumptions on $y$ or on $f$. In [2] one can find examples illustrating similar statements concerning various methods for solving ill-posed problems.

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# References

[1] V. Morozov, Methods of solving incorrectly posed problems, Springer Verlag, New York, 1984.

[2] A. G. Ramm, Inverse Problems, Springer, New York, 2005.

[3] A. G. Ramm, On unbounded operators and applications, (submitted)